J. Fluid Mech. (1967), *vol.* 28, *part* 1, *pp.* 183–193
Prtnted in Great Britain

A note on the expansion of turbulent wakes

By J. F. KEFFER
Department of Mechanical Engineering, University of Toronto, Canada

(Received 6 June 1966)

An analysis has been made of a two-dimensional turbulent wake which was subjected to a homogeneous strain. This was applied in such a way that the rate of growth of the wake tended to be accelerated but the main stream convective velocity remained constant. An attempt to describe the flow by classical self-preservation techniques was made. However, the results suggested that a detrainment of wake fluid would result. In lieu of this a more realistic approach was sought and it was found that in certain situations where the flow has had an opportunity to develop before the strain is applied, the growth of the wake in the strain field can be described successfully by a simple geometric distortion. It appears likely that in these cases the interaction between the external strain and the turbulent structure within the wake is negligible, the wake remaining essentially passive. Suggestions as to the mechanism of the turbulent mixing and transport under these conditions are made.

1. Introduction

The application of strain fields to free turbulence is of some intrinsic interest in the study of fluid dynamics. In certain cases it may increase our general understanding of some of the most pertinent and puzzling features of the flow. For example, by a properly oriented distortion, specific internal motions of say a turbulent wake, may be amplified, others will be attenuated or even completely suppressed. The features which emerge under these somewhat stimulated conditions can often shed light upon the internal structure of the normally developing flow.

Strains imposed by axisymmetric wind tunnel contractions have been investigated throughly in the past few decades and a fairly reasonable picture of the effects of these 'sudden' distortions has been reached. An interesting study of a homogeneous strain applied to an almost isotropic turbulent field was undertaken by Townsend (1954) in a special distorting duct. This was arranged to strain the flow at a constant rate in the two lateral directions, leaving the main stream mean velocity unchanged. Reynolds (1962) showed that with this type of distortion, a self-preserving solution could be obtained for a two-dimensional wake placed so that its width would tend to be compressed. However, since the unstrained wake is only asymptotically self-preserving, it is not too surprising that the experimental results did not agree with the self-preserving prediction. Keffer (1965) found that the orientation of the wake was such that the strain acted to amplify the large turbulent eddies within the flow. These eddies gained energy and became more intense as they moved through the distortion and eventually dominated the motion in the wake. The lateral transport of momentum and turbulent kinetic energy was almost entirely by these amplified eddies.

Because of these peculiarities a further examination of the two-dimensional wake was undertaken, but with the principle axes of lateral strain rotated through 90 degrees about the direction of the mean flow. The wake was thus subjected to a homogeneous strain which tended to accelerate its normal growth rate but again did not distort it along the mean flow axis. Measurements of velocity and length scales of the wake were obtained and comparisons made to a classical self-preservation analysis, which in the present case turns out to be inappropriate, and to a box distortion model, which appears to be successful in certain situations. These latest results are described in the following sections.

2. Description of the flow

A complete description of the wind tunnel is given by Townsend (1954). It is of the non-return type and consists of a two-dimensional channel, 6 by 24 in., the distorting test-section and a recovery section $24\frac{1}{4}$ by $6\frac{1}{4}$ in. The extra area represents the displacement effect of the boundary layers and is an attempt to main-

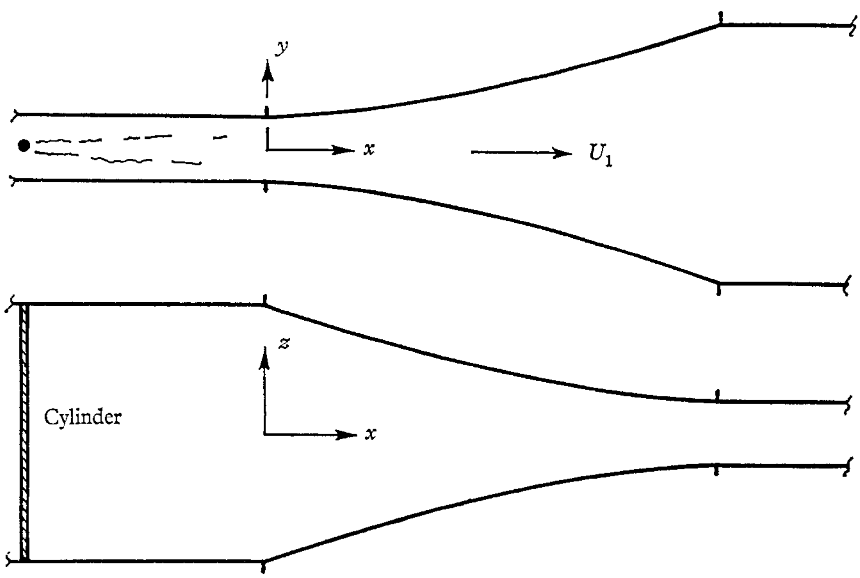

FIGURE 1. Distorting duct with cylinder 40 diameters upstream.

tain the mean flow velocity constant. The distorting duct (figure 1), is 40 in. long and has exponentially varying sides. The equations for the undisturbed streamlines within the duct are thus,

$$y = y_0 \exp\{a(x-x_0)\}, \tag{2.1}$$

$$z = z_0 \exp\{-a(x-x_0)\}, \tag{2.2}$$

where y_0 and z_0 are the duct dimensions at the beginning of the distortion x_0, and

a is a constant for the duct. The co-ordinate system for the flow is shown in figure 1. The convecting velocities are therefore,

$$U = U_1 = \text{constant}, \tag{2.3}$$

$$V = ayU, \tag{2·4}$$

$$W = -azU. \tag{2·5}$$

Circular cylinders were used as the wake-producing elements and were oriented symmetrically in the two-dimensional channel preceding the duct so that their axes were along the z-direction. The wake flow was thus expanded in the y-direction and contracted in the z-direction, but underwent no acceleration along the main-stream direction. The strain field is described by the tensor,

$$\alpha_{ij} = \partial U_i / \partial x_j = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & aU & 0 \\ 0 & 0 & -aU \end{Bmatrix}, \tag{2·6}$$

which satisfies the incompressibility condition, $\alpha_{ii} = 0$. Within the limits of the inhomogeneity of the tunnel turbulence, which in the present case was small compared to the wake turbulence, the strain was essentially uniform except for the boundary layers on the tunnel walls.

One of the few techniques which can be applied to turbulent shear flow analyses is an investigation of conditions for which self-preservation may exist. This method has been used successfully as a starting-point for many straightforward problems such as simple wakes, jets and certain classes of boundary layers. In the case of wake flows the self-preserving solution is only of asymptotic validity and requires the convecting flow to be of a larger order than the velocity defect. This condition is approximately satisfied after about 500 diameters of development. For the present situation all measurements were taken within 200 diameters. Nevertheless, it is perhaps interesting to attempt the analysis which can serve as a basis for comparison.

The argument follows in general the methods of Reynolds (1962) and Keffer (1965) for the compressed wake and is given in the appendix. The self-preserving relationships are found to be

$$u_0 \sim \beta^{\frac{1}{2}}, \quad l_0 \sim \beta^{\frac{1}{2}}, \tag{2.7, 2.8}$$

where u_0 and l_0 are characteristic velocity and length scales for the flow and β is the total strain which has been applied to the wake up to a point along the duct. The results indicate that both the defect velocity and wake width increase exponentially with distance downstream. The former condition would appear to be rather unlikely in the light of our present interpretation of free turbulent mixing. Experimental results given later suggest that the velocity scale can increase under certain conditions although the effect is slight.

The predicted increase in wake width would at first glance appear not unreasonable. However, the fluid velocity at the edge of the wake, say at l_0, must be at least $aU_1 l_0$. This is dictated by the convecting flow (2.4). On the other hand, the self-preserving analysis predicts a spread velocity of the turbulent front to be only,

$$U_1\, dl_0/dx = \tfrac{1}{2} a U_1 l_0. \tag{2.9}$$

This implies a negative rate of entrainment or 'detrainment' of the turbulent fluid, which within the present framework of our understanding of turbulent flows, has no reasonable interpretation. We must thus conclude that the assumptions required in the analysis to obtain a self-preserving solution are unrealistic, and attempt to find an alternative representation.

3. Experimental procedure

Two series of test were performed: (i) measurements of the mean velocity defect were taken for a $\frac{1}{2}$ in. diameter circular cylinder inserted at the entrance to the distorting duct and at a point 40 diameters upstream; (ii) further results

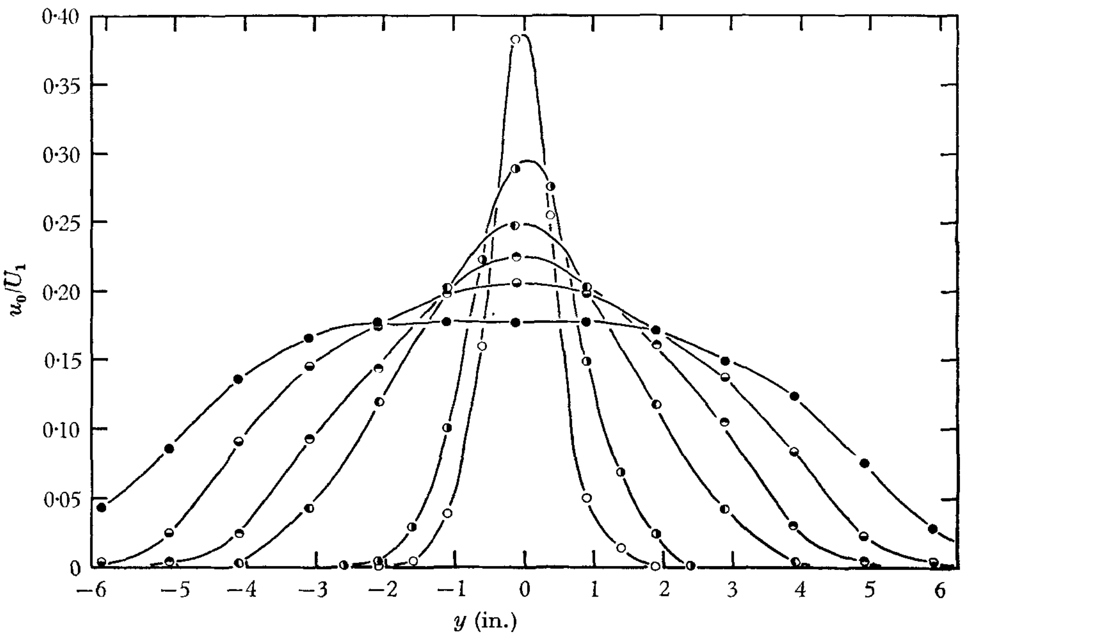

FIGURE 2. Mean velocity wake defect for $\frac{1}{2}$ in. diameter cylinder at distortion entrance; ○, $x-x_0$ = 5 in.; ◑, 10 in.; ◐, 20 in.; ◓, 25 in.; ◒, 30 in.; ●, 35 in.

were obtained with varying sized cylinders ($\frac{5}{16}$ and $\frac{3}{16}$ in.) placed at the beginning of the distortion. This enabled a comparison to be made of the effect of the strain imposed at different points in the wake development.

The mean velocity profiles were obtained with a small Pitot-static tube and sensitive micro-manometer. Qualitative turbulent intensities were used in some cases to evaluate the wake width. These served as a check on the functional variation of l_0 when the velocity defect was small. A sample of the mean velocity profiles is shown in figure 2. The velocity scale u_0 was defined as the maximum defect value. The length scale l_0 was chosen as the wake width at the one-half velocity point. The tunnel speed was held constant at 18 ft./sec so that the corresponding Reynolds numbers were 4630, 2890 and 1740 for the $\frac{1}{2}$, $\frac{5}{16}$ and $\frac{3}{16}$ in. diameter cylinders respectively. Thus viscous effects were not a significant factor, and the experimental range of Reynolds numbers was sufficiently small that wake flows were structurally similar for all cylinders.

4. Results and discussion

The length and velocity scales for the two series of tests are plotted in figures 3 and 4. The self-preserving variation, $\beta^{\frac{1}{2}}$, is included for the wake width, to emphasize the disagreement with the attempted analytical approach and as

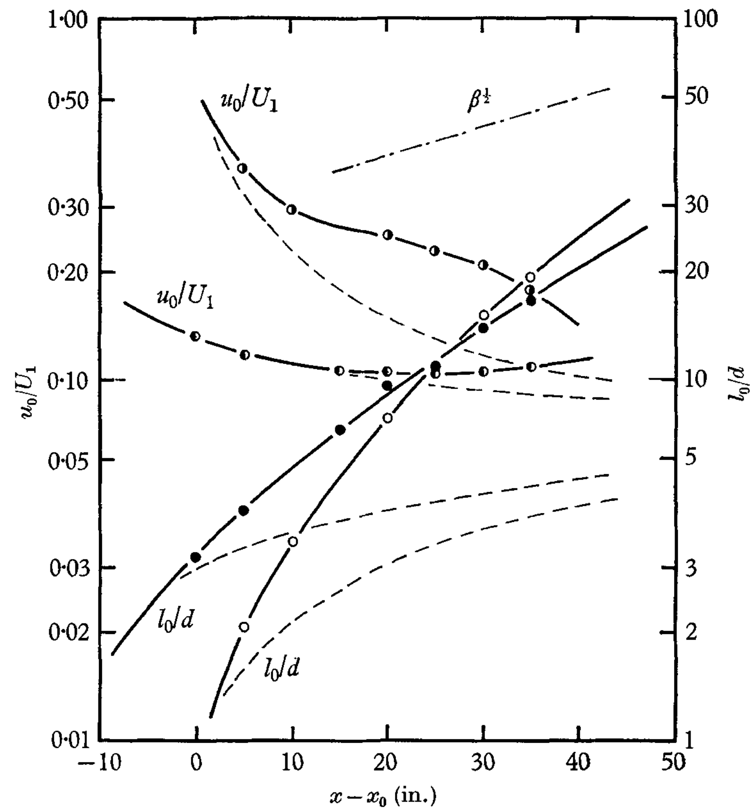

FIGURE 3. Wake scales for series (i) tests, $\frac{1}{2}$ in. cylinder; – – –, unstrained wake growth; —·—·—, $\beta^{\frac{1}{2}}$ variation for l_0; ◑ and ○, cylinder at $x-x_0 = 0$; ◐ and ●, cylinder at $x-x_0 = -20$ in.

expected, the wake growth is clearly greater than that suggested by relation (2·8). Detrainment of the turbulent fluid does not occur.

By comparing velocity scales for the strained and simple wakes it can be seen that the normal decay of the mean velocity defect has been retarded by the effect of the distortion. In figure 3 there is a suggestion that the velocity defect may even increase with the strain during the latter stages of the distortion. It would have been interesting to examine this effect for larger total strains, although experimental difficulties in extending the distorting section for the present series of tests were too great. For a duct length of much over 40 in., the boundary-layer growth would begin to influence the two-dimensional character of the flow significantly.

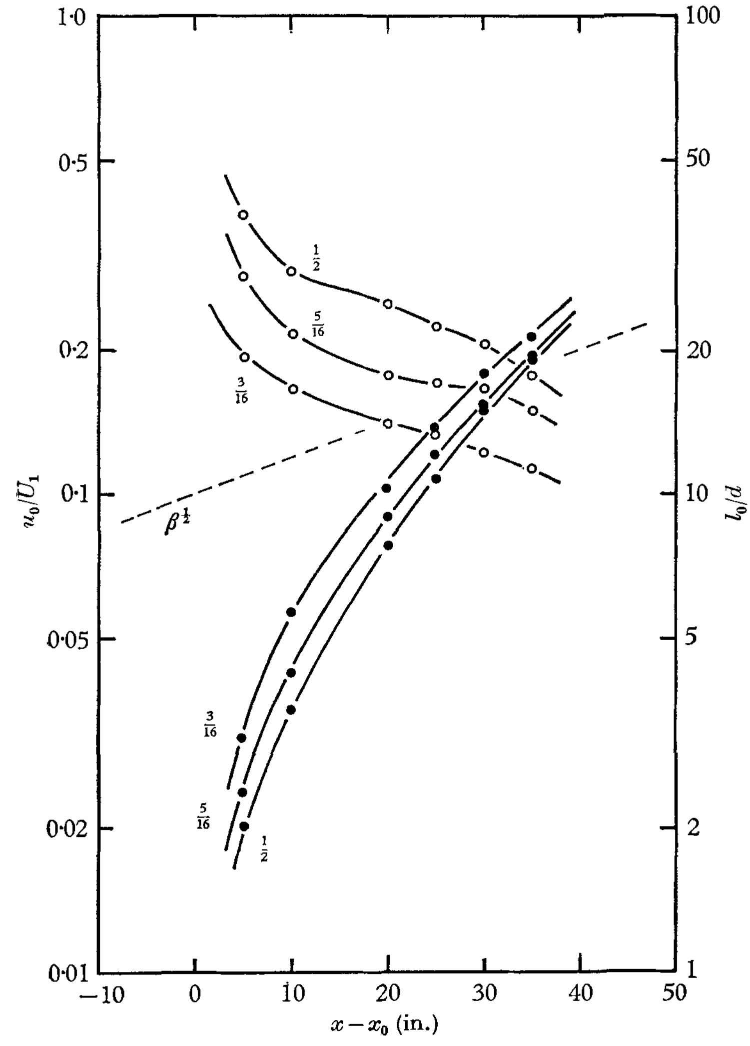

FIGURE 4. Wake scales for series (ii) tests; ○, velocity scale u_0/U_1; ●, width scale l_0/d.

As an alternative to the self-preservation theory one may take a more elementary approach by considering the variation of l_0 in a purely geometric sense. If the wake width can be considered to distort just as any other length in the system, for example the spacings between the streamlines, we have

$$(l_0)_{\text{dist}} = l_0 \times \beta. \tag{4·1}$$

This implies that the processes of normal wake growth and geometrical distortion are additive, i.e. no interaction takes place between the internal straining within the wake and the external strain field. This restriction is rather severe and it is doubtful whether a true box-type distortion of the wake could occur without some coupling.

To check this idea the normal unstrained wake growth (shown by the dashed lines in figures 3 and 4) was multiplied by the appropriate value of the total strain, β, and the results plotted in figures 5 and 6. From the first series of tests it is

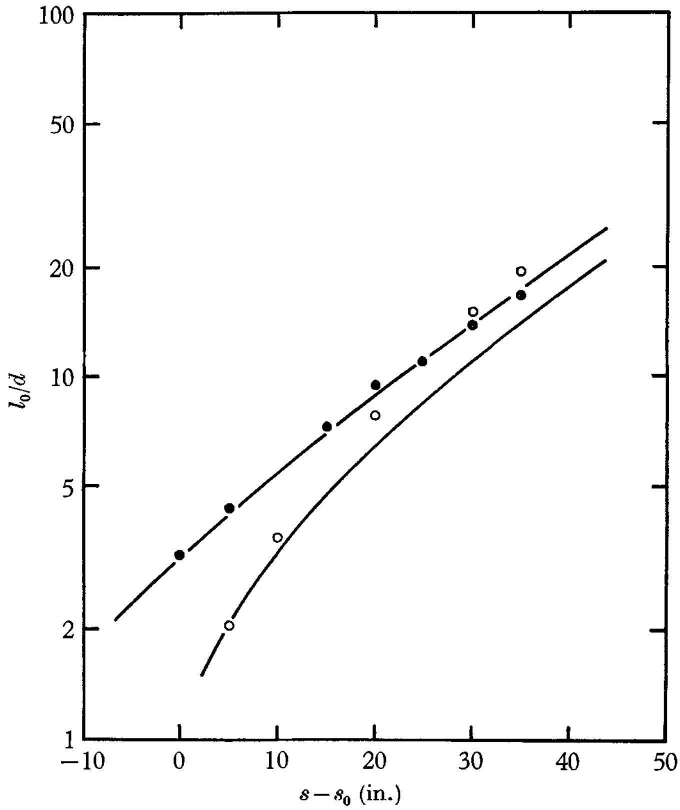

FIGURE 5. Comparison of experiment to geometric strain prediction for series (i) tests, $\frac{1}{2}$ in. cylinder; ——, $\beta l_0/d$; ○, cylinder at $x-x_0 = 0$; ●, cylinder at $x-x_0 = -20$ in.

evident that when the wake has undergone a period of development before the strain is imposed, the simple geometric strain theory predicts the behaviour extremely closely. If, however, the strain is applied at the inception of the wake, the growth rate exceeds (4.1) by an appreciable amount.

In the latter situation the relatively large and intense eddy motions which are present close to the cylinder as a result of the vortex shedding, have had no opportunity to diffuse before encountering the strain. The wake alignment is such that the strain field will attenuate their intensity but with a corresponding increase in their physical extent. This would result in a direct increase in the wake growth. For these conditions the restriction that there be no interaction between the wake turbulence and the external strain field, is no longer true. If, however, these vortices have had a chance to decay through the normal diffusive processes before encountering the strain, there would be no significant interaction and one would not expect this additional growth. It would appear that for the $\frac{1}{2}$ in. cylinder, 40 diameters of development are sufficient for this latter possibility to occur.

It is interesting to compare these results to the complementary situation (Keffer 1965), where the orientation of the wake permitted the strain field to amplify the intensity of the large eddy motions. When the strain was applied after

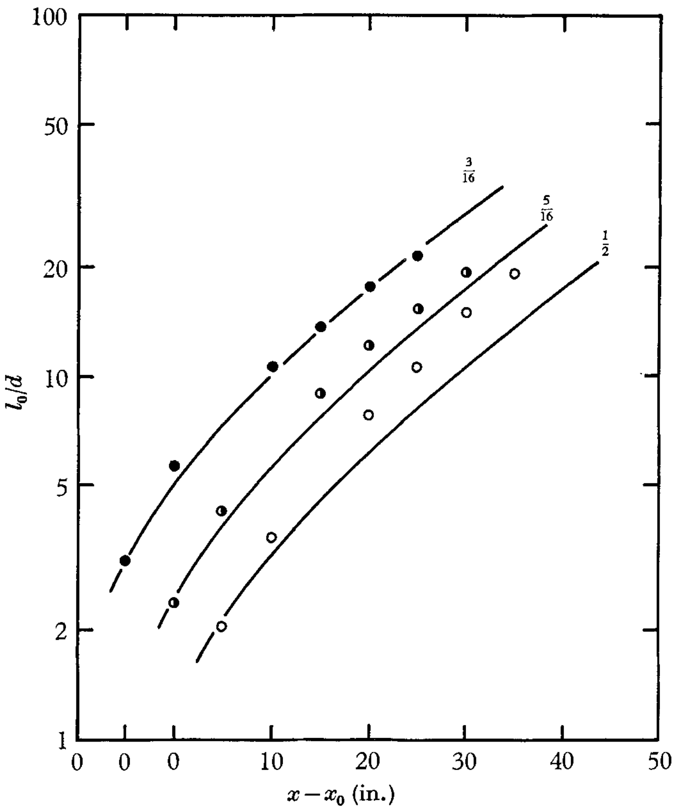

FIGURE 6. Comparison of experiment to geometric strain prediction for series (ii) tests (note successive shifts of zero on horizontal axis); ——, $\beta l_0/d$.

a similar period of development, the residual eddies gained energy directly from the strain field. This interaction increased the strength of the eddies to such an extent that they eventually dominated the structure of the flow and became the controlling influence in the lateral transport of momentum and turbulent kinetic energy. In contrast, the present situation appears to be entirely passive.

In the second series of tests, where the distortion is applied at the inception of the wake for varying cylinder diameters (figure 6), an additional aspect becomes apparent. Only the growth rate for the smallest cylinder ($\frac{3}{16}$ in.), is compatible with the prediction of wake growth via a pure geometric strain. It would thus appear that the size of the initial vortices is a contributing factor. If small enough, no interaction with the strain field takes place and the wake remains passive.

If too large, ($\frac{1}{2}$ and $\frac{5}{16}$ in. cylinders), interaction becomes significant and the growth rate is high.

If one were to apply the strain after the normal wake had reached the self-preserving state, the relation for which is

$$l_0 \sim x^{\frac{1}{2}}, \tag{4.2}$$

the geometric strain concept would predict a variation of the length scale as

$$(l_0)_{\text{dist}} = l_0 \times \beta \sim \beta x^{\frac{1}{2}}. \tag{4.3}$$

This result for a passive 'self-preservation' is functionally different from (2.4). Evidently classical self-preserving techniques can be applied only to rather simple situations and flows with external strain fields do not belong to this class.

The mean velocity distribution requires some comment. It is clear from figure 2 that the shape of the wake profiles are in no way self-similar. A distinct flattening appears, as the total strain increases, which tends to suggest that there is a marked homogeneous structure within the central core. This raises some interesting questions with regard to the internal mixing. If we accept the model of the large internal eddy structure suggested by Grant (1958), the effect of the strain on the interior of the flow may be pictured. These eddies, which are aligned at approximately 45 degrees to the mean flow axis, would be partially stretched by the strain and thereby increase in intensity. This would augment the normal wake mixing. The overall effect would become more pronounced as the total strain increased resulting in a corresponding large transfer of momentum throughout the core. Enhanced transfer would suppress the tendency for the velocity scale to increase with the total strain, an effect which is supported by the experimental results discussed earlier.

Production of turbulence is probably small in the core since the mean velocity gradient is almost zero. Reynolds stresses could be significant, however. For a more complete description, measurements of the turbulent quantities would be required and an assessment of the balance of turbulent energy made.

It is not clear what mechanism is most important in the entrainment of external fluid at the wake edge, a gradient type of diffusion, or the large laterally directed mixing jets which are significant in an unstrained wake. The latter effect would probably be small, however, since we have shown that these motions tend to be attenuated by the external strain. It is conceivable then that most of the transport would be by a gradient diffusion process, a conclusion in agreement with the passive wake characteristics which we have found.

This research was carried out in the Cavendish Laboratories of the University of Cambridge during tenure of a post-doctoral fellowship from the National Research Council of Canada. The author would like to thank Dr A. A. Townsend for the privilege of working in his laboratory and also express appreciation to Mr George Garner who constructed much of the equipment. The author is indebted to a referee who pointed out limitations to the self-preservation analysis.

Appendix

The self-preserving analysis is restricted to the central plane of symmetry, $z = 0$, and the flow is considered to be approximately two-dimensional. The simplified equations of motion, neglecting viscous effects are then

$$U\{\partial U/\partial x + ay\,\partial U/\partial y\} + \partial \overline{uv}/\partial y + \rho^{-1}\,\partial P/\partial x = 0, \tag{1}$$

$$a^2U^2y = -\rho^{-1}\,\partial P/\partial y. \tag{2}$$

When (2) is integrated with respect to the lateral direction y, and differentiated with respect to x, substitution into the first equation gives,

$$U_1\,\partial U/\partial x + a^2y^2U_1\,\partial U/\partial x + ayU_1\,\partial U/\partial y + \partial \overline{uv}/\partial y = 0, \tag{3}$$

where the convecting velocity within the wake U, has been replaced by U_1, the constant free stream velocity. This is sufficiently accurate for small deficit wakes.

In this form no self-preserving solution can be obtained. However, if the lateral wake scale is of a smaller order than the longitudinal dimension of the flow, the second term in (3) can be dropped. Experimentally, this condition is not met but, for the present, the assumption is made and the equation of motion becomes approximately,

$$U_1\{\partial U/\partial x + ay\,\partial U/\partial y\} + \partial \overline{uv}/\partial y = 0. \tag{4}$$

A second relationship, the momentum integral equation, can be obtained by integrating the fluxes across a suitably chosen control volume. With the above restrictions on the pressure gradient, the expression is approximately,

$$\beta^{-1}\int_y (U_1^2 - U_1\,U)\,dy = \text{const.}, \tag{5}$$

where β is the total strain applied by the duct up to a point along the flow,

$$\beta = \exp\{a(x-x_0)\}. \tag{6}$$

The self-preserving relationships for mean velocity and Reynolds stress are next introduced,

$$U = U_1 + u_0 f(\eta), \tag{7}$$

$$\overline{uv} = u_0^2 g_{12}(\eta), \tag{8}$$

where

$$\eta = y/l_0, \tag{9}$$

and u_0 and l_0 are the velocity and length scales for the wake respectively. Substituting (7) and (8) into (4), the differential equation of motion becomes,

$$\frac{U_1}{u_0}\left\{\frac{l_0}{u_0}\frac{du_0}{dx}f - \left(\frac{dl_0}{dx} - al_0\right)\eta f'\right\} + g_{12}' = 0, \tag{10}$$

which yields the restraints for self-reservation,

$$l_0 u_0^{-2}\,du_0/dx = \text{const.}, \tag{11}$$

and

$$u_0^{-1}\,dl_0/dx - al_0/u_0 = \text{const.} \tag{12}$$

Repeating this procedure with (5), we have to a first approximation,

$$\beta^{-1}U_1 u_0 l_0 \int_\eta f\,d\eta = \text{const.}, \tag{13}$$

and the third restraint is thus,

$$\beta^{-1}u_0 l_0 = \text{const.} \tag{14}$$

Solving for u_0 and l_0 from (11), (12) and (14), the functional variations of the wake scales can only be,

$$u_0 \sim \beta^{\frac{1}{2}} \sim \exp\{\tfrac{1}{2}a(x-x_0)\}, \tag{15}$$

$$l_0 \sim \beta^{\frac{1}{2}} \sim \exp\{\tfrac{1}{2}a(x-x_0)\}. \tag{16}$$

REFERENCES

GRANT, H. L. 1958 *J. Fluid Mech.* **4**, 149.
KEFFER, J. F. 1965 *J. Fluid Mech.* **22**, 135.
REYNOLDS, A. J. 1962 *J. Fluid Mech.* **13**, 333.
TOWNSEND, A. A. 1954 *Quart. J. Mech. Appl. Math.* **7**, 104.